Nonthermal nature of incipient extremal black holes

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17 March 2000; LaTeX-ed February 7, 2008

Abstract

We examine particle production from spherical bodies collapsing into extremal Reissner-Nordström black holes. Kruskal coordinates become ill-defined in the extremal case, but we are able to find a simple generalization of them that is good in this limit. The extension allows us to calculate the late-time worldline of the center of the collapsing star, thus establishing a correspondence with a uniformly accelerated mirror in Minkowski spacetime. The spectrum of created particles associated with such uniform acceleration is nonthermal, indicating that a temperature is not defined. Moreover, the spectrum contains a constant that depends on the history of the collapsing object. At first sight this points to a violation of the no-hair theorems; however, the expectation value of the stress-energy-momentum tensor is zero and its variance vanishes as a power law at late times. Hence, both the no-hair theorems and the cosmic censorship conjecture are preserved. The power-law decay of the variance is in distinction to the exponential fall-off of a nonextremal black hole. Therefore, although the vanishing of the stress tensor's expectation value is consistent with a thermal state at zero temperature, the incipient black hole does not behave as a thermal object at any time and cannot be regarded as the thermodynamic limit of a nonextremal black hole, regardless of the fact that the final product of collapse is quiescent.

PACS: 04.62.+v, 04.70.Dy, 04.20.Dw
Keywords: Black holes, Hawking radiation, moving mirrors.

1 Introduction

Extremal black hole solutions have long played a prominent role in black-hole thermodynamics. Early on, investigators realized that the zero surface gravity of extremal black

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holes, which implies zero Hawking temperature, makes them the natural equivalent of the zero temperature states in ordinary thermodynamics.

Nevertheless, the third law of black-hole dynamics [1, 2] states that the zero temperature state (the extremal black hole) is unattainable by means of a finite number of physical processes. The real status and meaning of this law is still subject of debate and investigation [3], but recently a point of view has emerged, according to which extremal black holes are thermodynamically different from the zero temperature limit of non-extremal ones [4, 5, 6, 7].

Over the past five years, advances in string theory [8] have also stimulated a resurgence of interest in extremal solutions. These have played a prominent role in both D-brane and supergravity calculations of black-hole entropy and these results seem to imply, contrary to what might be inferred from above, that the Bekenstein-Hawking relationship between entropy and area holds in the extremal case.

Semi-classical calculations, on the other hand, have thus far corroborated the conclusion implied by the third law, that the nature of extremal black holes intrinsically differs from that of nonextremal ones. In particular, such calculations predict a vanishing entropy for extremal black holes [6, 9, 10, 11, 12], contradicting the string-theory results.

Given the apparent incompatibility between the two approaches, and the fact that it might indicate some nontrivial issue in the low-energy limit of superstring theories, we try here to improve our understanding of the nature of extremal black holes from a semiclassical point of view. However, we shall not deal with the interpretation of the high-energy results in the present work, leaving this issue for future investigations.

The calculations cited above have mainly dealt with eternal black holes. It is thus unclear whether the thermodynamic discontinuity just mentioned applies to the case of black holes formed by collapse. For this reason we have decided to examine particle production by an "incipient" Reissner–Nordström (RN) black hole: A spherically symmetric collapsing charged body whose exterior metric is RN. In this paper we do not address the issue of actually constructing solutions of the Einstein equations that describe the collapse of charged configurations, because some simple solutions of this kind have already been found [13, 14, 15]. We emphasize that one of our main results is that the incipient extremal black hole does radiate in the early stages of the collapse. The fine-tuning which would then be required to produce the extremal solutions makes the former assumption of their existence highly nontrivial, because they would be extremely sensitive to effects such as the backreaction of the quantum radiation on the metric; we discuss these matters further in the conclusion. Nevertheless, for our purposes we assume that models can be found in which collapse leads to a black hole with $Q^2 = M^2$.

We approach the problem in standard fashion, modeling the collapse by a mirror moving in two-dimensional Minkowski spacetime [16]. The spectra resulting from the mirror's worldline will then be the same as that of the black hole, up to gray-body factors due to the nontrivial metric coefficients of RN spacetime and to the different dimensionality. However, to determine the appropriate worldline for the mirror one must employ coordinates that are regular on the event horizon, and although we find the tortoise coordinate r_* to be continuous at the $Q^2 = M^2$ limit, the usual Kruskal transformation fails there. Nevertheless, we provide a natural extension of Kruskal coordinates that is good for the $Q^2 = M^2$ case. The transformation cannot be explicitly inverted in terms of elementary functions, but is suitable for obtaining the asymptotic behavior for the collapsing

star. This leads us to consider a uniformly accelerated mirror in Minkowski spacetime, whose spectrum of created particles is nonthermal. We therefore conclude that incipient extremal RN black holes create particles with a nonthermal spectrum.

We find, moreover, that the spectrum's amplitude contains a constant that depends on the history of the collapsing object, apparently violating the no-hair theorems. However, the expectation value of the particles' stress-energy-momentum tensor is zero and its variance vanishes as a power law at late times. Consequently, particle creation dies out in the late stages of collapse, and is such that both the no-hair theorem and the cosmic censorship conjecture are preserved. One might argue that the zero value of the physical stress-energy-momentum tensor is consistent with a thermodynamic object at zero temperature. True enough, however, as we will see, the *approach* to zero of the stress tensor and its variance along with the non-Planckian spectrum indicate that the collapsing body acts like a thermal body at no time in its history. Therefore, although the final object is quiescent, it is improper to regard it as the zero temperature limit of a nonextremal black hole.

2 Kruskal-like coordinates for the extremal RN solution

Several textbooks in general relativity (see, e.g., Refs. [17, 18]) imply that Carter [19] found the maximal analytical extension of RN spacetime for $Q^2 = M^2$. In fact he made a very ingenious qualitative analysis without actually providing an analog of the Kruskal coordinates for the extremal case. Nevertheless, for our analysis it is essential to have such a coordinate transformation. For this reason we are going to retrace the steps leading to the maximal analytic extension of RN, paying close attention to the difference between the nonextremal and extremal situations.

The first step in the procedure is to define the so-called "tortoise" coordinate, which is then used to construct the Kruskal coordinates. We start with the usual form of the RN geometry,

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2\ , \tag{2.1}$$

where $\mathrm{d}\Omega^2$ is the metric on the unit sphere. The tortoise coordinate $r_*(Q, M)$ is given by

$$r_*(Q, M) = \int \frac{\mathrm{d}r}{(1 - 2M/r + Q^2/r^2)}\ . \tag{2.2}$$

Carrying out the integration yields, for the nonextremal case,

$$r_*(Q, M) = r + \frac{1}{2\sqrt{M^2 - Q^2}}\left(r_+^2\ln(r - r_+) - r_-^2\ln(r - r_-)\right) + \mathrm{const}, \tag{2.3}$$

where as usual $r_\pm = M \pm \sqrt{M^2 - Q^2}$.

Now, if we set $Q^2 = M^2$ in Eq. (2.2) *before* integrating, we find the "extremal" r_*:

$$r_*(M, M) = r + 2M\left(\ln(r - M) - \frac{M}{2(r - M)}\right) + \mathrm{const}. \tag{2.4}$$

Note that the coordinate $r_*(M,M)$ diverges only at $r = M$, but setting $Q^2 = M^2$ in $r_*(Q,M)$ appears to yield the indeterminate form 0/0. However, if we let $Q^2 = M^2(1-\epsilon^2)$, with $\epsilon \ll 1$, and work to first order in ϵ, it is straightforward to show that Eq. (2.3) does reduce to Eq. (2.4). Therefore r_* is continuous even at extremality.

Unfortunately, the Kruskal transformation itself breaks down at that point. The Kruskal transformation is

$$\left.\begin{array}{lll} \mathcal{U} = -\mathrm{e}^{-\kappa u} & \Leftrightarrow & u = -\dfrac{1}{\kappa}\ln(-\mathcal{U}) \\ \mathcal{V} = \mathrm{e}^{\kappa v} & \Leftrightarrow & v = \dfrac{1}{\kappa}\ln\mathcal{V} \end{array}\right\}, \tag{2.5}$$

where

$$\left.\begin{array}{l} u = t - r_* \\ v = t + r_* \end{array}\right\} \tag{2.6}$$

are the retarded and advanced Eddington-Finkelstein coordinates, respectively, and κ is the surface gravity. The latter is defined as

$$\kappa = \lim_{r\to r_+} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) = \frac{\sqrt{M^2 - Q^2}}{r_+^2}, \tag{2.7}$$

and vanishes for $Q^2 = M^2$. Therefore the Kruskal coordinates $\mathcal{U}$ and $\mathcal{V}$ become constant for any value of u and v and so the transformation (2.5) becomes ill-defined at that point.

We are nonetheless able to remedy this situation. Note that the Eddington-Finkelstein coordinates are constructed by adding or subtracting r_* to t, as in Eqs. (2.6) above. Now, for the extremal case, r_* is given by Eq. (2.4), which has the extra pole $M^2/(r-M)$ with respect to the strictly logarithmic dependence of the Schwarzschild and nonextremal RN cases (compare Eq. (2.3)). The simplest thing to do is define a function

$$\psi(\xi) = 4M\left(\ln\xi - \frac{M}{2\xi}\right) \tag{2.8}$$

and guess that a suitable generalization of the Kruskal transformation is

$$\left.\begin{array}{l} u = -\psi(-\mathcal{U}) \\ v = \psi(\mathcal{V}) \end{array}\right\}. \tag{2.9}$$

Note that $\psi'(\xi) = 4M/\xi + 2M^2/\xi^2 > 0$, always, and so ψ is monotonic; therefore (2.9) is a well-defined coordinate transformation. Note also that

$$r_*(M,M) = r + \frac{1}{2}\psi(r-M), \tag{2.10}$$

which means that near the horizon[1]

$$r_*(M,M) \sim \frac{1}{2}\psi(r-M)\ . \tag{2.11}$$

[1]Hereafter, for two functions f and g, we use the notation $f \sim g$ to mean $\lim f/g = 1$ in some asymptotic regime.

We can give our choice of ψ added motivation by noting that near the horizon Eq. (2.3) gives

$$r_*(Q,M) \sim \frac{1}{2\kappa}\,\ln(r-r_+)\ . \tag{2.12}$$

Thus we see that the function $\kappa^{-1}\ln(\cdots)$ that enters in the transformation (2.5) from the Kruskal to the Eddington-Finkelstein coordinates, is just twice the one which gives a singular contribution to $r_*(Q,M)$ at $r=r_+$. Our extension (2.9) is therefore analogous to the Kruskal transformation (2.5): We choose ψ as the part of r_* that is singular at $r=r_+$, a procedure that should work in other, similar situations.

For (2.9) to be a good coordinate extension, the new coordinates $\mathcal{U}$ and $\mathcal{V}$ must be regular on the event horizon, $\mathcal{H}$. This will be the case if the metric after the coordinate transformation is singular only at $r=0$. For the extremal case the metric in terms of u and v reads

$$\mathrm{d}s^2 = -\left(1-\frac{M}{r}\right)^2 \mathrm{d}u\,\mathrm{d}v + r^2\,\mathrm{d}\Omega^2\ . \tag{2.13}$$

Written in terms of the "Kruskal-like" coordinates $\mathcal{U}$ and $\mathcal{V}$, the metric (2.13) assumes the form

$$\mathrm{d}s^2 = -\frac{(r-M)^2}{r^2}\psi'(-\mathcal{U})\psi'(\mathcal{V})\mathrm{d}\mathcal{U}\mathrm{d}\mathcal{V} + r^2\,\mathrm{d}\Omega^2\ . \tag{2.14}$$

This line element apparently is degenerate at $\mathcal{H}$; if so the transformation is ill-defined there. We now show, however, that the factor $(r-M)^2$ is actually killed and that the transformation is regular at $r=M$.

At $\mathcal{H}$ the coordinate v is always finite and so asymptotically we have $t\sim -r_*$. Therefore $u\sim -2r_*\sim -\psi(r-M)$ where the last approximation follows from Eq. (2.11). The inverse transformation yields

$$\mathcal{U} = -\psi^{-1}\left(-u\right) \sim -\psi^{-1}\left(\psi(r-M)\right) = -\left(r-M\right)\ . \tag{2.15}$$

Then, from the expression for ψ' given above we have near the horizon

$$\psi'(-\mathcal{U}) \sim \frac{4M}{r-M} + \frac{2M^2}{(r-M)^2} \sim \frac{2M^2}{(r-M)^2}\ . \tag{2.16}$$

Furthermore, since $\mathcal{V}$ is everywhere nonzero and finite then $\psi'(\mathcal{V})$ is regular there. Now it is easy to see that the form taken by the metric (2.14) is asymptotically

$$\mathrm{d}s^2 \sim -\frac{2M^2}{r^2}\psi'(\mathcal{V})\mathrm{d}\mathcal{U}\mathrm{d}\mathcal{V} + r^2\,\mathrm{d}\Omega^2\ . \tag{2.17}$$

The $(r-M)^2$ in the numerator of Eq. (2.14) is killed by the $(r-M)^2$ in the denominator of Eq. (2.16). Consequently, $\mathcal{U}$ and $\mathcal{V}$ are good Kruskal-like coordinates.

Notice that the coordinates u and v defined by the transformation (2.5) do not tend to those given by (2.9) as $Q^2\to M^2$. This is related to the fact that the maximal analytic extensions of RN spacetime are qualitatively different in the two cases [18], and is another evidence of the discontinuous behaviour mentioned in the Introduction.

3 Asymptotic worldlines

With the result of the previous section in hand we are now able to construct late-time asymptotic solutions for the incipient extremal black hole. Our goal is to find an equation for the center of the collapsing star (in the coordinates u and v) that is valid at late times. Equation (2.6) gives u and v outside the collapsing star, but the center of the star, of course, is in the interior. We must therefore extend u and v to the interior. Since u and v are null coordinates, representing out- and in-going light rays, respectively, the extension can be accomplished almost trivially by associating any event on the interior of the star with the u and v values of the light rays that intersect at this event.

The most general form of the metric for the interior of a spherically symmetric star can be written as

$$\mathrm{d}s^2 = \gamma(\tau,\chi)^2(-\mathrm{d}\tau^2 + \mathrm{d}\chi^2) + \rho(\tau,\chi)^2\,\mathrm{d}\Omega^2\ , \tag{3.1}$$

where γ and ρ are functions that can be chosen to be regular on the horizon. From the coordinates τ and χ we can construct interior null coordinates $U = \tau - \chi$ and $V = \tau + \chi$, which will also be regular on the horizon. The center of the star can be taken at $\chi = 0$, in which case $V = U$ and $\mathrm{d}V = \mathrm{d}U$ there (see Fig. 1). Because the Kruskal coordinates

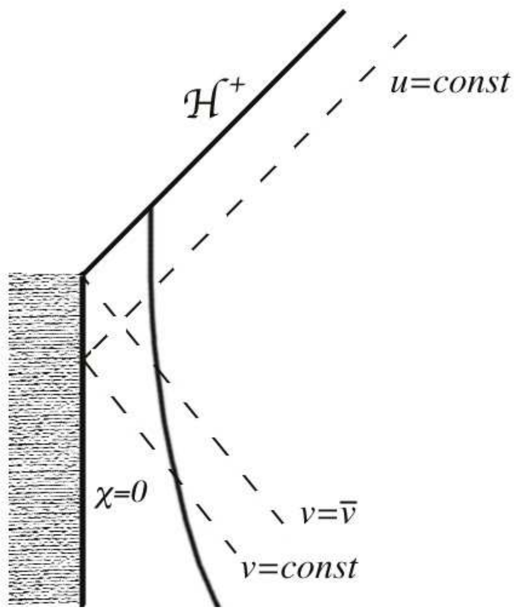

Figure 1: A representation of gravitational collapse in null coordinates. The portion of spacetime beyond the event horizon $\mathcal{H}^+$ is not shown.

$\mathcal{U}$ and $\mathcal{V}$ are regular everywhere as well, they can be matched to U and V. In particular, if two nearby outgoing rays differ by $\mathrm{d}U$ inside the star, then they will also differ by $\mathrm{d}U = \beta(\mathcal{U})\mathrm{d}\mathcal{U}$, with β a regular function, outside. By the same token, since V and v are regular everywhere, we have $\mathrm{d}V = \zeta(v)\mathrm{d}v$, where ζ is another regular function. In fact, if we consider the last ray $v = \bar{v}$ that passes through the center of the star before the formation of the horizon, then to first order $\mathrm{d}V = \zeta(\bar{v})\mathrm{d}v$, where $\zeta(\bar{v})$ is now constant.

We can write near the horizon

$$\mathrm{d}U = \beta(0)\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}u}\mathrm{d}u \ . \tag{3.2}$$

Since for the center of the star $\mathrm{d}U = \mathrm{d}V = \zeta(\bar{v})\mathrm{d}v$, this immediately integrates to

$$\zeta(\bar{v})(v - \bar{v}) = \beta(0)\mathcal{U}(u) = -\beta(0)\psi^{-1}(-u) \sim -2\beta(0)\frac{M^2}{u} \ . \tag{3.3}$$

The last approximation follows from Eq. (2.8) where $\xi \sim \psi^{-1}(-2M^2/\xi)$ near the horizon.

Thus the late-time worldline for the center of the star is, finally, represented by the equation[2]

$$v \sim \bar{v} - \frac{A}{u} \ , \quad u \to +\infty \ , \tag{3.4}$$

where $A = 2\beta(0)M^2/\zeta(\bar{v})$ is a positive constant that depends on the details of the internal metric and consequently on the dynamics of collapse.

We first note that the worldline (3.4) differs from the one resulting from the collapse of a nonextremal object, which would be of the form (see e.g. [16, 21, 22])

$$v \sim \bar{v} - B\mathrm{e}^{-\kappa u} \ , \quad u \to +\infty \ . \tag{3.5}$$

One immediately wonders, then, if our result can be recovered in the case of nonextremal black holes by simply going to a higher order approximation for the asymptotic worldline of the center of the collapsing star. It is easy to see that this is not the case. Recall that in the Kruskal coordinates $\mathcal{U}$ and $\mathcal{V}$, the horizon is located at $\mathcal{U} = 0$. Say the worldline of the center of the star crosses the horizon at some $\mathcal{V} = \overline{\mathcal{V}}$. Let us expand $\mathcal{V}(\mathcal{U})$ in a Taylor series around $\mathcal{U} = 0$ such that $\mathcal{V} = \overline{\mathcal{V}} + \alpha_1\mathcal{U} + \alpha_2\mathcal{U}^2$. The term $\alpha_1\mathcal{U} \propto \mathrm{e}^{-\kappa u}$ is the usual one found for the thermal case and $\alpha_2\mathcal{U}^2$ is the correction. However, $\mathcal{U}^2 \propto \mathrm{e}^{-2\kappa u}$ and so this term is also a constant for extremal incipient black holes. In fact corrections are constant to arbitrary order. The extremal worldline in no sense, therefore, represents a limit of the nonextremal case but implies a real discontinuity in the asymptotic behavior of the collapsing object.

Equations (3.4) and (3.5) contain the constants A and B, which are determined by the dynamics of collapse. In the nonextremal case, it is known that no measurement performed at late times can be used to infer the value of B, thus enforcing the no-hair theorem. In particular, the spectrum of Hawking radiation depends only on the surface gravity κ. It is natural to ask whether a similar statement holds true also for extremal black holes. This point will be analyzed in the following sections.

4 Bogoliubov coefficients

Let us now consider a test quantum field in the spacetime of an incipient extremal RN black hole. For the sake of simplicity, and without loss of generality, we can restrict our

[2]This result was also obtained by Vanzo [20] for a collapsing extremal thin shell, but without considering a coordinate extension. Our method is completely general and shows that Eq. (3.4) follows only from the kinematics of collapse and the fact that the external geometry is the extremal RN one.

analysis to the case of a hermitian, massless scalar field ϕ. Instead of dealing with a black hole proper, we consider a two-dimensional Minkowski spacetime with a timelike boundary [16, 21]. This spacetime is described by null coordinates (u, v) and the equation governing the boundary is the same that describes the worldline of the center of the star, say $v = p(u)$.[3] At the centre of the star the ingoing modes of ϕ become outgoing, and vice versa; this translates into the requirement that on the spacetime boundary there is perfect reflection, or that $\phi(u, p(u)) \equiv 0$. Hence, "mirror": The timelike boundary in Minkowski spacetime is traced out by a one-dimensional moving mirror for the field ϕ.

In general, for a worldline $v = p(u)$ one has $\mathrm{d}\tau = \sqrt{p'(u)}\,\mathrm{d}u$, where τ is the proper time along the worldline. From this and the fact that the acceleration for the trajectory in two-dimensional Minkowski spacetime is $a = \frac{1}{2}\sqrt{p''(u)^2/p'(u)^3}$, one can easily check that Eqs. (3.4) and (3.5) yield $a^2 = 1/A$ and $a^2 = \kappa \mathrm{e}^{\kappa u}/(4B)$, respectively. Thus we see that an incipient extremal black hole is modeled at late times by a uniformly accelerated mirror; for nonextremal black holes the acceleration of the mirror increases exponentially with time. In both cases the mirror's worldline has a null asymptote $v = \bar{v}$ in the future, while it starts from the timelike past infinity i^- at $t = -\infty$.

Without loss of generality, one can assume that the mirror is static for $t < 0$. A suitable worldline is then

$$p(u) = u\Theta(-u) + f(u)\Theta(u)\ , \tag{4.1}$$

where Θ is the step function, defined as

$$\Theta(\xi) = \begin{cases} 1 & \text{if} \quad \xi \geq 0\ , \\ 0 & \text{if} \quad \xi < 0\ , \end{cases} \tag{4.2}$$

and $f(u)$ is a function with the asymptotic form (3.4). In order for the worldline to be C^1, $f(u)$ must be such that $f(0) = 0$ and $f'(0) = 1$. To simplify calculations, it is convenient to choose $f(u)$ hyperbolic at all times [21], i.e.,

$$f(u) = \sqrt{A} - \frac{A}{u + \sqrt{A}}\ , \tag{4.3}$$

which coincides with the function in the right hand side of Eq. (3.4), up to a (physically irrelevant) translation of the origin of coordinates.

Due to the motion of the mirror, one expects that the In and Out vacuum states will differ, leading to particle production whose spectrum depends on the function $p(u)$. In our case, because the mirror worldline has a null asymptote $v = \bar{v}$ in the future but no asymptotes in the past, the explicit forms of the relevant In and Out modes for ϕ are easily shown to be

$$\phi_\omega^{(\mathrm{in})}(u, v) = \frac{\mathrm{i}}{\sqrt{4\pi\omega}} \left(\mathrm{e}^{-\mathrm{i}\omega v} - \mathrm{e}^{-\mathrm{i}\omega p(u)}\right) \tag{4.4}$$

and

$$\phi_\omega^{(\mathrm{out})}(u, v) = \frac{\mathrm{i}}{\sqrt{4\pi\omega}} \left(\mathrm{e}^{-\mathrm{i}\omega u} - \Theta\left(\bar{v} - v\right) \mathrm{e}^{-\mathrm{i}\omega q(v)}\right)\ , \tag{4.5}$$

[3] In Refs. [16, 21, 23] the function p is defined somewhat differently. For a generic shape $x = z(t)$ of the boundary, one first defines a quantity τ_u through the implicit relation $\tau_u - z(\tau_u) = u$. Then, the function is defined as $p(u) = 2\tau_u - u$, which is exactly the phase of the outgoing component of the In modes, and $v = p(u)$ is just the equation for boundary's worldline.

where $q(v) = p^{-1}(v)$ and $\omega > 0$. The spectrum of particles created in such a scenario is known, although, to our knowledge, no one has pointed out the correspondence to the formation of extremal black holes. However, since the result is something of a textbook case, we here merely summarize the main steps; for details, see e.g. Ref. [21], p 109.

The In and Out states of ϕ can be related by the Bogoliubov coefficients:

$$\alpha_{\omega\omega'} = \left(\phi_\omega^{(\mathrm{out})}, \phi_{\omega'}^{(\mathrm{in})}\right) = -\mathrm{i}\int_0^{+\infty} \mathrm{d}x \left[\phi_\omega^{(\mathrm{out})}(u,v)\overleftrightarrow{\partial_t}\,\phi_{\omega'}^{(\mathrm{in})}(u,v)^*\right]_{t=0} ; \tag{4.6}$$

$$\beta_{\omega\omega'} = -\left(\phi_\omega^{(\mathrm{out})}, \phi_{\omega'}^{(\mathrm{in})*}\right) = \mathrm{i}\int_0^{+\infty} \mathrm{d}x \left[\phi_\omega^{(\mathrm{out})}(u,v)\overleftrightarrow{\partial_t}\,\phi_{\omega'}^{(\mathrm{in})}(u,v)\right]_{t=0} . \tag{4.7}$$

The spectrum of created particles is given by the expectation value of the "out quanta" contained in the In state, $\langle 0,\mathrm{in}|N_\omega^{(\mathrm{out})}|0,\mathrm{in}\rangle$. In terms of the Bogoliubov coefficients this spectrum is

$$\langle N_\omega\rangle = \int_0^{+\infty} \mathrm{d}\omega'\,|\beta_{\omega\omega'}|^2 , \tag{4.8}$$

where $\langle N_\omega\rangle$ is shorthand for $\langle 0,\mathrm{in}|N_\omega^{(\mathrm{out})}|0,\mathrm{in}\rangle$.

With the choice (4.3), one can compute Bogoliubov coefficients that are appropriate in the asymptotic regime $t \to +\infty$. Performing the integrals in Eqs. (4.6) and (4.7) gives [21]

$$\alpha_{\omega\omega'} \approx \mathrm{i}\frac{\sqrt{A}}{\pi}\mathrm{e}^{-\mathrm{i}\sqrt{A}(\omega+\omega')}K_1(2\mathrm{i}(A\omega\omega')^{1/2}) , \tag{4.9}$$

$$\beta_{\omega\omega'} \approx \frac{\sqrt{A}}{\pi}\mathrm{e}^{\mathrm{i}\sqrt{A}(\omega-\omega')}K_1(2(A\omega\omega')^{1/2}) , \tag{4.10}$$

where K_1 is a modified Bessel function, shown in Fig. 2. For argument z, $K_1(z) \sim 1/z$ for $z \to 0$, and $K_1(z) \sim \sqrt{\pi/(2z)}\,\mathrm{e}^{-z}$ when $z \to +\infty$ [24].

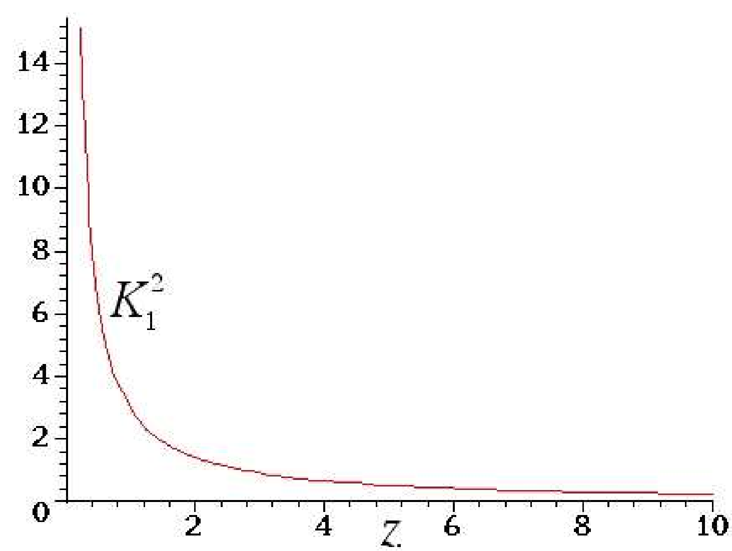

Figure 2: Plot of the modified Bessel function $K_1(z)$, squared.

We emphasize that Eqs. (4.9) and (4.10) *do not* correspond to a full evaluation of the integrals in Eqs. (4.6) and (4.7), but only take into account the contribution for $x \approx \sqrt{A}$, i.e., from the mirror worldline at $u \to +\infty$. This is the only part of the Bogoliubov coefficients that can be related to particle creation by an incipient black hole, because any other contribution corresponds to particles created much earlier, and depends therefore on the arbitrary choice of $p(u)$ in the non-asymptotic regime.[4] Clearly, since $\langle N_\omega \rangle \neq 0$, there is particle creation by the incipient extremal RN black hole.[5]

Due to the $1/(\omega\omega')$ in the asymptotic form of $|\beta_{\omega\omega'}|^2$, the spectrum (4.8) diverges at low frequencies. The divergence in ω' has the same origin as the one that appears in the case of nonextremal black holes, where

$$|\beta_{\omega\omega'}|^2 = \frac{1}{2\pi\omega'} \left(\frac{1}{\mathrm{e}^{2\pi\omega/\kappa} - 1} \right) . \tag{4.11}$$

These Bogoliubov coefficients also contain a logarithmic divergence in ω', which is due to the evaluation of the mode functions at $u = +\infty$ and for that reason can be interpreted as an accumulation of an infinite number of particles after an infinite time. The divergence can be removed, however, as Hawking suggested [26] by the use of wave packets instead of plane-wave In states; this has the effect of introducing a frequency cutoff.

Contrary to what happens in the nonextremal case, $\langle N_\omega \rangle$ is not a Planckian distribution and therefore the spectrum of created particles is nonthermal. Thus, the notion of temperature is undefined. This result supports the view that an extremal black hole is not the zero temperature limit of a nonextremal one. However, it would be premature to base these conclusions only on the basis of Eq. (4.8), because the Bogoliubov coefficients tell us only that particles are created *at some time* in the late stages of collapse, which does not necessarily mean that such creation takes place at a steady rate. In the next two sections we refine our conclusions through an analysis of the stress-energy tensor of the quantum field.

5 Preservation of cosmic censorship

Equations (4.8) and (4.10) indicate that an incipient extremal RN black hole creates particles with a spectrum that depends on the constant A. These results immediately raise two problems. First, since particle creation leads to black hole evaporation, it seems that (some version of) the cosmic censorship conjecture could be violated. Indeed, emission of neutral scalar particles implies a decrease in M, while Q remains constant; evidently, a transition to a naked singularity ($Q^2 > M^2$) should take place. Second, the dependence of the spectrum on A, which in turn depends on the details of collapse, raises the possibility of getting information about the collapsing object through measurements performed at late times, a contradiction of the no-hair theorems.

[4] There has been some discussion in the literature [25] about whether the calculation of the Bogoliubov coefficients by Fulling and Davies [16, 21] is correct. We find that their approximations are valid in the asymptotic regime of interest to us.

[5] This result is only apparently in contradiction with the analysis performed in Ref. [20], where it is claimed that there is no emission of neutral scalar particles. In fact, such a conclusion was derived for a massive field in the ultrarelativistic limit, and agrees with the exponential behaviour of K_1 at large values of ω.

We consider the first problem. The luminosity of the black hole, the rate of change of M, is given by the flux of created particles at infinity, or the T_{uu} component of the stress-energy tensor. Wu and Ford [23] have recently provided the expectation value of T_{uu} for the case of a moving boundary in two-dimensional Minkowski spacetime:

$$\langle :T_{uu}: \rangle = \frac{1}{4\pi} \left(\frac{1}{4} \left(\frac{p''}{p'} \right)^2 - \frac{1}{6} \frac{p'''}{p'} \right) . \tag{5.1}$$

Inserting the form (4.1) of p, with f given by Eq. (4.3), into Eq. (5.1), one gets

$$\langle :T_{uu}: \rangle = \frac{1}{24\pi\sqrt{A}}\, \delta(u) . \tag{5.2}$$

Thus, the only nonvanishing contribution to $\langle :T_{uu}: \rangle$ is due to the transition from uniform to hyperbolic motion that takes place at $t = 0$. For the discussion of incipient black holes only the behaviour for $u \to +\infty$ is relevant, and so this feature is uninteresting. On the other hand, in the hyperbolic regime $\langle :T_{uu}: \rangle$ vanishes identically. (It is also straightforward to check from Eq. (5.1) that, conversely, a hyperbolic worldline is the only one with nonzero acceleration that leads to $\langle :T_{uu}: \rangle = 0$.)

The result shows that the flux due to an incipient extremal black hole vanishes asymptotically at late times. Consequently, extremal black holes do not loose mass,[6] and cosmic censorship is preserved. However, the nonzero value of $\beta_{\omega\omega'}$ clearly shows that there *is* particle creation during collapse. Cosmic censorship has apparently been rescued only at the price of introducing a paradox, namely: Particles are created *and* their flux has zero expectation value. How can these two statements be simultaneously true?

This puzzling situation has been extensively discussed in the context of particle emission from a uniformly accelerating mirror [16, 21, 29]. Fulling and Davies [16] explain the net zero energy flux in the presence of nonzero $\beta_{\omega\omega'}$ by a special cancellation of the created modes via quantum interference, which is due to contributions from the coefficients $\alpha_{\omega\omega'}$. In the Appendix, we analyze this issue further by examining the response function of a detector.

6 Preservation of the no-hair theorem

We now turn to the second of the problems mentioned earlier: Given that the spectrum contains the constant A, do extremal black holes violate the no-hair theorems? The result $\langle :T_{uu}: \rangle = 0$ suggests an escape — in spite of the nonzero value of $\langle N_\omega \rangle$, no radiation is actually detected. However, this resolution raises new questions. If no radiation is detected, how can one claim that the black hole emits anything at all? Is the radiation observable? How should one then interpret $\langle N_\omega \rangle$?

It is premature to claim that no radiation is detected only on the basis of $\langle :T_{uu}: \rangle = 0$, because there could be other nonvanishing observables from which one might infer the presence of quanta. A straightforward calculation shows that the expectation values of

[6] Here, we assume that luminosity is simply related to $\langle :T_{uu}: \rangle$, which amounts to assuming the validity of the semiclassical field equation $G_{\mu\nu} = 8\pi \langle :T_{\mu\nu}: \rangle$ [27]. This, however, might not be a good approximation when ϕ is in a state with strong correlations (see, e.g., [28] and references therein).

T_{vv} and T_{uv} are also zero. However, let us examine the variance ΔT_{uu} of the flux. Wu and Ford [23] have also recently given the following expression for $\langle :T^2_{uu}: \rangle$ in the case of a minimally coupled, massless scalar field in two-dimensional Minkowski spacetime with a timelike boundary described by the equation $v = p(u)$:

$$\langle :T^2_{uu}: \rangle = \frac{1}{(4\pi)^2}\left(-\frac{4p'^2}{(v-p(u))^4} + \frac{3}{16}\left(\frac{p''}{p'}\right)^4 - \frac{1}{4}\frac{p'''}{p'}\left(\frac{p''}{p'}\right)^2 + \frac{1}{12}\left(\frac{p'''}{p'}\right)^2\right) . \qquad (6.1)$$

If one ignores the so-called cross terms [23], this coincides with the variance ΔT_{uu} (because in our case $\langle :T_{uu}: \rangle = 0$). With p given by Eqs. (4.1) and (4.3), Eq. (6.1) gives, for $u > 0$,

$$\langle :T^2_{uu}: \rangle = -\frac{A^2}{4\pi^2\left(A + (v-\bar{v})\,u\right)^4} \sim -\frac{A^2}{4\pi^2\left(v-\bar{v}\right)^4 u^4} . \qquad (6.2)$$

Thus, in spite of the fact that the expectation value of the flux vanishes identically, its statistical dispersion does not, but its value becomes smaller and smaller and tends to zero in the limit $u \to +\infty$. Hence, although one could in principle infer the value of the constant A by measuring the quantity ΔT_{uu} at late times, such measurements will become more and more difficult as ΔT_{uu} decreases according to Eq. (6.2). This damping is of course reminiscent of the familiar damping of perturbations, which prevents one from detecting by late-time measurements the details of an object that collapses into a black hole [30]. And so, monitoring ΔT_{uu} does not lead to a violation of the no-hair theorem, because no trace of A will survive in the limit $u \to +\infty$.

This discussion shows only that no violation of the no-hair theorem can be detected by measuring the variance in the energy flux. The possibility remains that other types of measurement could allow one to find out the value of A. If, however, $\Delta T_{\mu\nu} \to 0$ for $u \to +\infty$, then the random variable $T_{\mu\nu}$ must tend to its expectation value, i.e., to zero. This means that, asymptotically, the properties of the field are those of the vacuum state. Consequently, all local observables will tend to their vacuum value.

Although extremal black holes obey the no-hair theorems, the *way* in which cosmic baldness is enforced differs from the nonextremal situation. Consider again the variance of the flux. Inserting the function p for nonextremal incipient black holes (see Eq. (3.5)) into Eqs. (5.1) and (6.1), one gets $\langle :T_{uu}: \rangle = \kappa^2/(48\pi)$ and

$$\langle :T^2_{uu}: \rangle = \frac{1}{(4\pi)^2}\left(\frac{\kappa^4}{48} - \frac{4\kappa^2 B^2 \mathrm{e}^{-2\kappa u}}{\left(v-\bar{v}+B\mathrm{e}^{-\kappa u}\right)^4}\right) \sim \frac{\kappa^4}{768\pi^2} - \frac{\kappa^2 B^2 \mathrm{e}^{-2\kappa u}}{4\pi^2\left(v-\bar{v}\right)^4} . \qquad (6.3)$$

Contrary to the extremal case, the “nonextremal” variance ΔT_{uu} tends not to zero as $u \to +\infty$, but to the value $\kappa^2\sqrt{2}/(48\pi)$, which corresponds to thermal emission; this is sufficient to guarantee that no information about the details of collapse is conveyed. Furthermore, the approach to this value is exponentially fast, while for the extremal configuration the decay obeys only a power law.

7 Conclusions

We have found a simple generalization of Kruskal coordinates that allows us to examine the behavior of incipient, extremal RN black holes. Although the coordinate transformation we employ is not invertible in terms of elementary functions, it makes possible the

explicit calculation of the asymptotic form of the worldline for the center of the collapsing object. Borrowing well-known results from quantum field theory in the presence of moving boundaries, we concluded that an incipient extremal black hole emits particles with a nonthermal spectrum, which contains a constant that depends on the details of gravitational collapse.

At first sight, this result seems to imply that the cosmic censorship conjecture and the no-hair theorem are both violated by extremal black holes. Closer scrutiny reveals that the flux of emitted radiation vanishes identically, and in the limit $t \to +\infty$ any measurement of local observables gives results indistinguishable from those in the vacuum state. This is not incompatible with a nonzero spectrum, which is not a local quantity and tells us only that particles are created at some time during collapse (not necessarily at $t = +\infty$). Thus, extremal black holes are not pathological in this respect.

However, there are several clearly defined senses in which nonextremal and extremal black holes differ. Information lost to an external observer depends on the rate at which the statistical dispersion of the flux approaches its value for $t \to +\infty$. In the nonextremal case, the dispersion goes to zero exponentially fast, Eq. (6.3), whereas for an incipient extremal black hole it follows a slower power law, given by Eq. (6.2). More importantly, for $Q^2 \to M^2$, Eq. (6.2) is not the limit of Eq. (6.3)[7]. One cannot therefore, consider quantum emission by an incipient extremal black hole to be the limiting case of emission by a nonextremal black hole. In particular, although at $t = +\infty$ a black hole with $Q^2 = M^2$ is totally quiescent it would be incorrect to consider it as the thermodynamic limit of a nonextremal black hole, that is, an object at zero temperature. Indeed, the quantum radiation emitted by an incipient extremal black hole is not characterized by a temperature at any time during collapse. Whereas incipient nonextremal black holes have a well defined thermodynamics, this is not true for extremal holes, and they should be considered as belonging to a different class. This result suggests that any calculations that implicitly rely on a smooth limit in thermodynamic quantities at $Q^2 = M^2$ are suspect, if not incorrect. Our conclusions, of course, are just pertinent to incipient black holes; extending them to eternal black holes seems plausible, but requires care. (Even at the classical level, eternal black holes must be regarded as fundamentally different from those deriving from collapse, because the global structure of spacetime differs in the two cases.)

We close the paper drawing an analogy between the exotic subject of particle production by extremal black holes and a well-known piece of ordinary physics. The divergence of the particle spectrum $\langle N_\omega \rangle$ is reminiscent of the infrared catastrophe typical of QED, which manifests itself, for instance, in the process of bremsstrahlung (see, e.g., Ref. [31], pp. 165–171). However, the infrared divergence in the bremsstrahlung cross section produces no observable effect, because it is canceled by analogous terms coming from radiative corrections. (Thanks to the Bloch-Nordsieck theorem, this cancellation is effective to all orders of perturbation theory.) One may well wonder whether the $\omega = 0$ singularity in our spectrum is similarly fictitious and could thus be removed by analogous techniques.

For the mirror this is possible, in principle, if one allows momentum transfer from the

[7] Since A and B do not depend on u by definition, the only case that admits a continuous limit is the one in which $A = 0$. This cannot happen, because it would correspond to a null worldline for the centre of the star. Another apparent possibility, that $B \propto 1/\kappa$ so that κB is constant in the limit $\kappa \to 0$, is not viable, because the right hand sides of Eqs. (6.2) and (6.3) would still have different functional dependences on u. We thank Freeman Dyson for pointing out the issue to one of us.

field ϕ to the mirror, although such a calculation is beyond the scope of the present paper. (See Ref. [32] for a model that includes recoil.) But, whatever the answer to the mirror problem might be, it does not seem that one could transplant it in any straightforward way to the case of an incipient black hole. Indeed, taking recoil into account would amount to admitting that backreaction *is* important and that the test-field approximation is never valid. Thus, the whole subject would have to be reconsidered within an entirely different framework.

In connection with the possible — and crucial — relevance of backreaction, it is important to stress one aspect of our results. We have seen in Sec. 5 that if $Q^2 = M^2$ at the onset of the hyperbolic worldline (3.4), it will remain so and the cosmic censorship conjecture is preserved. However, the fact that mass loss from an incipient extremal black hole is zero *only* in the late hyperbolic stage seems to imply that an enormous fine tuning is required in order to produce an extremal object by means of gravitational collapse. In fact, an object that is extremal from the start of its collapse might be unstable with respect to the transition to a configuration with $Q^2 > M^2$. Such a transition would be triggered by quantum emission in the early phases of collapse, when $p(u)$ has not yet assumed its hyperbolic form. This raises the question of how, in presence of quantum radiation, the formation of a naked singularity is prevented (e.g., by the emission of charged particles) and the cosmic censorship conjecture preserved.

Note added Simultaneously with this work Anderson, Hiscock and Taylor [34] have demonstrated that for static RN geometries, zero-temperature black holes cannot exist if one considers spacetime perturbations due to the back reaction and quantum fields.

Acknowledgements It is a pleasure to thank J. Almergren for support in drawing Fig. 1 and F. Belgiorno for helpful remarks on a first draft of the paper. S.L. thanks R. Parentani for stimulating discussions. T.R. would like to acknowledge the hospitality of S.I.S.S.A., where this work was carried out.

Appendix: Detecting radiation from a uniformly accelerated mirror

In Sec. 5 we mentioned the apparently paradoxical situation in which nonzero particle production (as shown by nonvanishing Bogoliubov coefficients $\beta_{\omega\omega'}$) is accompanied by zero energy flux (vanishing expectation value of the stress-energy-momentum tensor.) Discussions about such issues are often phrased in terms of ideal detectors [21, 33]. Although our arguments in the body of the paper are based solely on the behavior of the stress-energy-momentum tensor, we can gain some additional insight into the "paradox" by considering the response of a simple monopole detector on a geodesic worldline $v = u + 2x_0$, with $x_0 = \text{const}$, in two-dimensional Minkowski spacetime.

We are interested in computing the detector response function per unit time, defined as

$$\mathcal{R}(E) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{d}\tau \int_{-T}^{T} \mathrm{d}\tau' \, \Theta(E) \mathrm{e}^{-\mathrm{i}E(\tau-\tau')} D^+(u(\tau), v(\tau); u(\tau'), v(\tau')) \, , \tag{A.1}$$

where D^+ is the Wightman function of the scalar field in the In vacuum, $u(\tau) = \tau - x_0$, $v(\tau) = \tau + x_0$, and E is the excitation energy of the detector. (Note that $E \geq 0$, which is

automatically enforced by the presence of the step function $\Theta(E)$ in the right hand side of Eq. (A.1).) In terms of the In modes, D^+ has the form

$$D^+(u,v;u',v') = \int_{-\infty}^{+\infty} \mathrm{d}\omega\, \Theta(\omega)\phi_\omega^{(\mathrm{in})}(u,v)\phi_\omega^{(\mathrm{in})}(u',v')^* \,, \tag{A.2}$$

where we have extended the integration range to $-\infty$, by introducing the step function $\Theta(\omega)$.

Since the definition of $\mathcal{R}(E)$ involves an integration over time from $-\infty$ to $+\infty$, in the case of a mirror worldline of the type (4.1) it will get contributions corresponding to the nonzero flux (like, e.g., the one at $u = 0$ when $f(u)$ is given by Eq. (4.3)). These we regard as spurious, because we are really interested in clarifying the relationship between zero flux and nonzero spectrum in the hyperbolic regime. For this reason, let us consider a mirror worldline which is hyperbolic at all times, say $p(u) = -A/u$ for $u > 0$, for which there can be no such spurious contributions to $\mathcal{R}(E)$.

The worldline $p(u) = -A/u$ has a null asymptote in the past, thus

$$\phi_\omega^{(\mathrm{in})}(u,v) = \frac{\mathrm{i}}{\sqrt{4\pi\omega}}\left(\mathrm{e}^{-\mathrm{i}\omega v} - \Theta(u)\mathrm{e}^{-\mathrm{i}\omega p(u)}\right) \,. \tag{A.3}$$

On substituting Eq. (A.3) into Eq. (A.2), we have

$$D^+(u,v;u',v') = F_1(v,v') + F_2(u,v') + F_3(v,u') + F_4(u,u') \,, \tag{A.4}$$

where:

$$F_1(v,v') = \frac{1}{4\pi}\int_{-\infty}^{+\infty} \mathrm{d}\omega\, \frac{\Theta(\omega)}{|\omega|}\, \mathrm{e}^{-\mathrm{i}\omega(v-v')} \,; \tag{A.5}$$

$$F_2(u,v') = -\frac{1}{4\pi}\Theta(u)\int_{-\infty}^{+\infty} \mathrm{d}\omega\, \frac{\Theta(\omega)}{|\omega|}\, \mathrm{e}^{\mathrm{i}\omega(v'-p(u))} \,; \tag{A.6}$$

$$F_3(v,u') = -\frac{1}{4\pi}\Theta(u')\int_{-\infty}^{+\infty} \mathrm{d}\omega\, \frac{\Theta(\omega)}{|\omega|}\, \mathrm{e}^{-\mathrm{i}\omega(v-p(u'))} \,; \tag{A.7}$$

$$F_4(u,u') = \frac{1}{4\pi}\Theta(u)\Theta(u')\int_{-\infty}^{+\infty} \mathrm{d}\omega\, \frac{\Theta(\omega)}{|\omega|}\, \mathrm{e}^{-\mathrm{i}\omega(p(u)-p(u'))} \,. \tag{A.8}$$

Correspondingly, $\mathcal{R}(E)$ can be split into four parts: $\mathcal{R}(E) = \mathcal{R}_1(E) + \mathcal{R}_2(E) + \mathcal{R}_3(E) + \mathcal{R}_4(E)$.

The terms $\mathcal{R}_1(E)$, $\mathcal{R}_2(E)$, and $\mathcal{R}_3(E)$ can be computed straightforwardly, by using the formal identities

$$\lim_{T\to+\infty}\int_{-T}^{T} \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\xi\tau} = 2\pi\delta(\xi) \tag{A.9}$$

and

$$\frac{1}{|E|}\,\Theta(E)\Theta(-E) = 2\delta(E) \,, \tag{A.10}$$

the latter being easily established by considering the sequence of functions $(|E|+\epsilon)^{-1}\Theta(E+\epsilon)\Theta(-E+\epsilon)$ in the limit $\epsilon \to 0$. We get $\mathcal{R}_1(E) = -2\mathcal{R}_2(E) = -2\mathcal{R}_3(E) = \delta(E)$, so the first three contributions to $\mathcal{R}(E)$ sum to zero.

The computation of $\mathcal{R}_4(E)$ is cleaner if one works in dimensionless variables, such as $\widetilde{E}=\sqrt{A}\,E$, $\tilde{\omega}=\sqrt{A}\,\omega$, $\tilde{\tau}=\tau/\sqrt{A}$. The identity

$$\int_0^{+\infty} \mathrm{d}\tilde{\omega}\,\frac{\mathrm{e}^{-\mathrm{i}\tilde{\omega}(\xi-\mathrm{i}0)}}{\tilde{\omega}} = -\ln(\xi-\mathrm{i}0)+I-\mathrm{i}\,\frac{\pi}{2}\,, \tag{A.11}$$

where I is the divergent quantity

$$I=\int_0^{+\infty} \mathrm{d}\tilde{\omega}\,\frac{\cos\tilde{\omega}}{\tilde{\omega}}\,, \tag{A.12}$$

together with the properties of the logarithm, allows us to write

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathrm{d}\tilde{\omega}\,\frac{\Theta(\tilde{\omega})}{|\tilde{\omega}|}\,\exp\left(-\mathrm{i}\tilde{\omega}\frac{\tilde{\tau}-\tilde{\tau}'}{(\tilde{\tau}-\tilde{x}_0)\,(\tilde{\tau}'-\tilde{x}_0)}\right) &= \int_{-\infty}^{+\infty} \mathrm{d}\tilde{\omega}\,\frac{\Theta(\tilde{\omega})}{|\tilde{\omega}|}\,\mathrm{e}^{-\mathrm{i}\tilde{\omega}(\tilde{\tau}-\tilde{\tau}')} \\ -\int_{-\infty}^{+\infty} \mathrm{d}\tilde{\omega}\,\frac{\Theta(\tilde{\omega})}{|\tilde{\omega}|}\,\mathrm{e}^{-\mathrm{i}\tilde{\omega}(\tilde{\tau}-\tilde{x}_0)} &- \int_{-\infty}^{+\infty} \mathrm{d}\tilde{\omega}\,\frac{\Theta(\tilde{\omega})}{|\tilde{\omega}|}\,\mathrm{e}^{\mathrm{i}\tilde{\omega}(\tilde{\tau}'-\tilde{x}_0)}+2I\,.\end{aligned} \tag{A.13}$$

In this expression we have replaced one of the quantities $I-\mathrm{i}\pi/2$ with its complex conjugate by simultaneously changing the sign in one of the exponents. This manipulation is allowed by the fact that, since $\tilde{\tau}-\tilde{x}_0$ and $\tilde{\tau}'-\tilde{x}_0$ can never become negative in F_4, their logarithms are always real. Note that the resulting expression agrees with the property of $\mathcal{R}(E)$ of being a real quantity.

We can thus write $\mathcal{R}_4(E)=\mathcal{R}_{41}(E)+\mathcal{R}_{42}(E)+\mathcal{R}_{43}(E)+\mathcal{R}_{44}(E)$. Using the formal relations

$$\lim_{\widetilde{T}\to+\infty}\int_{\tilde{x}_0}^{\widetilde{T}} \mathrm{d}\tilde{\tau}\,\mathrm{e}^{\mathrm{i}\xi\tilde{\tau}} = \pi\delta(\xi)+\mathrm{i}\mathrm{e}^{\mathrm{i}\,\xi\tilde{x}_0}\,\mathrm{P}\left(\frac{1}{\xi}\right)\,, \tag{A.14}$$

and

$$\lim_{\widetilde{T}\to+\infty}\frac{1}{\widetilde{T}}\int_{\tilde{x}_0}^{\widetilde{T}} \mathrm{d}\tilde{\tau}\,\mathrm{e}^{\mathrm{i}\xi\tilde{\tau}} = \Theta(\xi)\Theta(-\xi)\,, \tag{A.15}$$

together with Eq. (A.10), we get $\mathcal{R}_{41}(E)=\delta(E)/4$, $\mathcal{R}_{42}(E)+\mathcal{R}_{43}(E)=-\delta(E)/2$, and $\mathcal{R}_{44}(E)=I\delta(E)/4$. Finally, since I is divergent we can write

$$\mathcal{R}(E)=\frac{I}{4}\,\delta(E)\,. \tag{A.16}$$

Thus, we have essentially a delta function peaked at zero energy. Now, $\mathcal{R}(E)$ is related to a quantum mechanical probability and so this result means that, for any value $E>0$ of the energy, no matter how small, the detector has probability 1 of making a transition of amplitude smaller that E and probability 0 of detecting particles of higher energy. (Of course, this does not mean it will *never* make transitions with $E>0$; only, these take place with probability 0.) The reason for this behaviour is evidently the divergence in the spectrum as $\omega\to 0$. Of course, the detector does not gain energy during such a "detection" — in fact, one can say that there is no detection at all. This is compatible with the zero value of the flux.

Hence one might be tempted to call the particles emitted by a mirror in hyperbolic motion "phantom radiation": Because only arbitrarily soft particles would be registered by the detector with any nonzero probability, there would be no chance to determine the

spectrum $\langle N_\omega \rangle$. The question arises therefore, whether there is any way to screen our detector from this overwhelming flux of soft quanta.

One might think to act on the selectivity of the detector by using a two level system that requires at least a minimal energy to switch. Unfortunately, the detection of the infinite tail of soft quanta corresponds to the "transition" from the ground state to the ground state, and there is obviously no way to forbid this process. The detector cannot be forbidden to not switch!

So also the analysis of the response function of a detector seems to prove that the radiation from uniformly accelerated mirrors (and extremal incipient black holes) is in some sense like the apple in Dante's purgatory: We can see it with our mind but we shall never have it in our hands...

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